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Group analysis of structure equations for stars in radiative and convective equilibrium

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Abstract

It is proposed to use the Lie group theory of symmetries of differential equations to investigate the system of equations describing a static star in a radiative and convective equilibrium. It is shown that the action of an admissible group induces a certain algebraic structure in the set of all solutions, which can be used to find a family of new solutions. We have demonstrated that, in the most general case, the equations admit an infinite parameter group of quasi-homologous transformations. We have found invariants of the symmetry groups which correspond to the fundamental relations describing a physical characteristic of the stars such as the Hertzsprung–Russell diagram or the mass–luminosity relation. In this way we can suggest that group invariants have not only purely mathematical sense, but their forms are closely associated with the basic empirical relations.

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1. Introduction

A modern theory of star interiors and their evolution was founded on the works of Emden, Lane, Ritter and Kelvin [1]. They studied equilibrium configurations of polytropic and isothermic gaseous spheres. The important results were obtained by Rudzki [2] who introduced the notion of homologous transformations. Eddington [3] described the effectiveness of radiation transport of energy in a star and built the standard model of stellar structure. Chandrasekhar [1] formulated the homologous theorem known as the Stromgren theorem. Schwarzschild [4] described the thermonuclear processes as a source of energy of the stars. Some properties of stellar structure can be expressed in terms of the Lie group theory of symmetries to obtain a deeper insight into the structure of solution of equation for stars in radiative and connective equilibrium [5, 6].

One of the main problems in the group analysis of differential equations is the investigation of properties of the group admissible by the differential equations structure. In the set of all

solutions the action of an admissible group induces a certain algebraic structure which can be used to find a family of new solutions from the known ones [7–12].

In the present work, we apply the Lie group theory to investigate group properties of the system of four structure equations describing Newtonian static stars in radiative and convective equilibrium. Two of these equations, namely the hydrostatic equilibrium and the mass continuity equation, were investigated by Collins from the point of view of group theory [13].

As is well known from Stromgren's theorem [14], new solutions, for such a system of equations, can be obtained from the known ones through the homologous transformations. The new solutions describe new configurations with different masses, radii and chemical compositions (the so-called homologous stars). We generalize this result by introducing the notion of quasi-homologous stars, i.e., stars whose equations of state admit quasi-homology symmetries. The homologous stars are a special case of the quasi-homologous ones. At the present stage of our investigation, this should be treated as a purely mathematical result, although it cannot be excluded that the obtained dependences between luminosity and temperature, mass and temperature and so on could be employed in a manner similar to that done by Stromgren [14] to fully interpret the Hertzsprung–Russell (HR) diagram.

The energy of the star radiated away from its surface is generally replenished from reservoirs situated in the very hot central region. The transport of energy is possible due to the existence of a non-vanishing temperature gradient in the star. The transfer can occur mainly via radiation, conduction and convection. In any case, photons, nuclei and electrons are exchanged between hotter and cooler parts where the direction of the energy flow is determined by the temperature gradient.

The problem of application of symmetry group in the context of equations coming from astrophysics was considered in our previous papers [5, 6]. This paper is a continuation of new quasi-homologous symmetries which are present in the structural equations for stars with convective transport.

2. Mathematical background

In the present work we consider a differential equation system of the following form:

$$\frac{du^i}{dx} = f^i(x, u^1, \dots, u^m) \quad i = 1, \dots, m. \quad (1)$$

We consider point transformations generated by the infinitesimal operator

$$X = \xi(x, u^1, \dots, u^m) \frac{\partial}{\partial x} + \sum_{i=1}^m \eta^i(x, u^1, \dots, u^m) \frac{\partial}{\partial u^i}. \quad (2)$$

For the infinitesimal operator X there are m independent invariants which are solutions to the following system (if we assume the analyticity of a field)

$$\frac{dx}{\xi(x, u^1, \dots, u^m)} = \frac{du^1}{\eta^1(x, u^1, \dots, u^m)} = \dots = \frac{du^m}{\eta^m(x, u^1, \dots, u^m)}. \quad (3)$$

The point transformation generated by X is called homologous if $\xi = ax$ and $\eta^i = g^i u^i$ where a, g^i ($i = 1, \dots, m$) are constants. Of course, if $X \equiv \lambda^s(u) \partial_s$ then the finite transformation of symmetry $u \rightarrow \bar{u}$ is given as a solution of equation $d\bar{u}^s/d\tau = \lambda^s(u)$ with the initial conditions $\bar{u}^s(\tau = 0) = u^s$, $s = 1, \dots, m$. It can easily be seen that system (1) is similarity invariant in this case (for a discussion, see [15]).

A natural extension of this special case leads us to the notion of quasi-homologous transformations $\xi = \xi(x), \eta^i = \eta^i(u^i), i = 1, \dots, m$. System (1) admits the infinitesimal operator (2) if and only if the following condition is satisfied

$$\frac{\partial \eta^i}{\partial x} + \sum_{j=1}^m \left(\frac{du^j}{dx} \frac{\partial \eta^i}{\partial u^j} - \frac{du^i}{dx} \frac{du^j}{dx} \frac{\partial \xi}{\partial x} - \frac{du^i}{dx} \frac{\partial \xi}{\partial x} \right) - X(f^i) = 0. \tag{4}$$

Condition (4) tells us whether the symmetry operator X is admitted by system (1) [7]. It is easily seen that, in the case of a quasi-homologous transformation, equation (4) assumes the form

$$\frac{d\eta^i}{du^i} - \frac{d\xi}{dx} = X(\ln f^i). \tag{5}$$

Now let us consider the space of independent variable x , dependent u^α and its first derivatives, say $(u^\alpha)'$ ($\alpha = 1, \dots, m$). The action of the Lie group G of the point transformation in the space (x, u) can be extended to the space (x, u, u') . On the other hand an s th order (ordinary or partial) differential equation $F(x, u(x), u'(x), \dots, u^s(x)) = 0$ defines a certain manifold \mathcal{M} in the space (x, u, u', \dots, u^s) . We say that F is invariant with respect to the group G , provided that the manifold \mathcal{M} is a fixed point with respect to the s th extension of G , i.e., $G^s(\mathcal{M}) = \mathcal{M}$. In terms of an infinitesimal operator it means that

$$X^s F|_{F=0} = 0 \tag{6}$$

where for the case of first-order differential equations

$$\begin{aligned} X^s &= X + \xi_i^\alpha(x, u, u') \frac{\partial}{\partial u_i^\alpha} & u_i^\alpha &= \frac{\partial u^\alpha}{\partial x^i} \\ \xi_i^\alpha &= D_i(\eta^\alpha) - u_j^\alpha D_i(\xi^j) & D_i &= \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha}. \end{aligned}$$

The prolonged operators, which are generators of symmetry of equations in the space (x, u) , form the structure of the Lie algebra of a fundamental group.

The different method of construction of a partial solution of the system without knowledge of the fundamental group is based on searching some subgroup which is called a similarity group, for which the finite transformations are

$$\bar{u}^i = a^i u^i \quad \bar{x}^\alpha = a^{n+\alpha} x^\alpha \quad i = 1, \dots, n \quad \alpha = 1, \dots, m$$

and the Lie algebra is determined by the operator

$$X_i \equiv u^i \frac{\partial}{\partial u^i} \quad X_{n+\alpha} = x^\alpha \frac{\partial}{\partial x^\alpha}$$

where all coefficients—dilatation coefficient— a^i, \dots, a^{n+m} are positive. It is easy to find such a subgroup because all form

$$a^j = (a^1)^{m_1 j} (a^2)^{m_2 j} \dots (a^k)^{m_k j} \quad j = k + 1, \dots, N.$$

The invariants of this subgroup are given by

$$\Pi = (u^1)^{\alpha_1} (u^2)^{\alpha_2} \dots (u^{N-1})^{\alpha_{N-1}} (u^N)^{\alpha_N}$$

where $\alpha_1, \dots, \alpha_N$ are determined from algebraic equations. There is strictly a connection between the dimensional analysis of equations and in describing its fundamental group.

3. Quasi-homologous transformations of structure equations with radiative transport

The structure of the equation for a Newtonian star is as follows [4]

$$\frac{dp}{dr} = -\frac{GM\rho}{r^2} \quad \text{hydrostatic equilibrium} \quad (7)$$

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad \text{mass continuity} \quad (8)$$

$$\frac{dL}{dr} = 4\pi r^2 \epsilon(\rho, T) \quad \text{thermal equilibrium} \quad (9)$$

either

$$\frac{dT}{dr} = -\frac{3}{16\pi ac} \frac{\rho}{r^2} \frac{L}{T^3} \kappa(\rho, T) \quad \text{radiative equilibrium} \quad (10)$$

or

$$\frac{dT}{dr} = \frac{\Gamma_2 - 1}{\Gamma_2} \frac{T}{p} \frac{dp}{dr} = \Gamma \frac{T}{p} \frac{dp}{dr} \quad \text{adiabatic convective equilibrium} \quad (11)$$

where M is the mass within the sphere of radius r , ρ is the density, p is the pressure, L is the luminosity at the surface of the sphere of radius r , T is the temperature, ϵ is the energy generation rate, κ is the opacity, G is the gravitational constant, c is the velocity of light and a is the Stefan–Boltzmann constant.

First we consider the case of radiative equilibrium. The equation for energy transport through the stellar material can be written as a condition for the temperature gradient necessary for the required energy flow. It supplies the next basic equation for the stellar structure. Assuming the equation of state in the form $p = p(\rho, T)$ we can rewrite equation (7) in the more convenient form

$$\frac{d\rho}{dr} = \left(\frac{\partial p}{\partial \rho}\right)^{-1} \left(-GM + \frac{3}{16\pi ac} \frac{L}{T^3} \kappa \frac{\partial p}{\partial T}\right) \frac{\rho}{r^2}. \quad (12)$$

Now, we look for the symmetry transformations of equations (8)–(10) and (12) generated by the operator

$$X = \xi(r) \frac{\partial}{\partial r} + \eta^1(\rho) \frac{\partial}{\partial \rho} + \eta^2(M) \frac{\partial}{\partial M} + \eta^3(L) \frac{\partial}{\partial L} + \eta^4(T) \frac{\partial}{\partial T}. \quad (13)$$

If we denote

$$f = -GM + \frac{3}{16\pi ac} \frac{L}{T^3} \kappa \frac{\partial p}{\partial T}$$

equations (5) for quasi-homologous transformations take the form

$$\begin{aligned} \frac{d\eta^1}{d\rho} - \frac{d\xi}{dr} &= -\frac{2\xi}{dr} + \frac{\eta^1}{\rho} - \eta^1 \frac{\partial}{\partial \rho} \left(\ln \frac{\partial p}{\partial \rho}\right) + \frac{\eta^1}{f} \left[\frac{3}{16\pi ac} \frac{L}{T^3} \frac{\partial}{\partial \rho} \left(\kappa \frac{\partial p}{\partial T}\right)\right] - \frac{G\eta^2}{f} \\ &+ \frac{\eta^3}{f} \left(\frac{3}{16\pi ac} \frac{\kappa}{T^3} \frac{\partial p}{\partial T}\right) + \frac{\eta^4}{f} \left[-\frac{\partial}{\partial T} \left(\ln \frac{\partial p}{\partial \rho}\right) + \frac{3L}{16\pi ac} \frac{\partial}{\partial T} \left(\frac{\kappa}{T^3} \frac{\partial p}{\partial T}\right)\right] \end{aligned} \quad (14)$$

$$\frac{d\eta^2}{dM} - \frac{d\xi}{dr} = \frac{2\xi}{r} + \frac{\eta^1}{\rho} \quad (15)$$

$$\frac{d\eta^3}{dL} - \frac{d\xi}{dr} = \frac{2\xi}{r} + \frac{\eta^1}{\rho} + \frac{\eta^1}{\rho} \frac{\partial \epsilon}{\partial \rho} + \frac{\eta^4}{\epsilon} \frac{\partial \epsilon}{\partial T} \quad (16)$$

$$\frac{d\eta^4}{dT} - \frac{d\xi}{dr} = -\frac{2\xi}{r} + \frac{\eta^1}{\rho} + \frac{\eta^1}{\kappa} \frac{\partial \kappa}{\partial \rho} + \frac{\eta^3}{L} + \eta^4 \left(-\frac{3}{T} + \frac{1}{\kappa} \frac{\partial \kappa}{\partial T} \right). \quad (17)$$

Since the right-hand side of equation (15) depends only on ρ , $\eta^1 = \alpha_1 \rho$, where α_1 is constant. It is easy to verify, by the same argument and by substitution into equations (14)–(17), that also

$$\eta^2 = \alpha_2 M \quad \eta^3 = \alpha_3 L \quad \xi = \frac{1}{3}(\alpha_2 - \alpha_1)r$$

where α_2, α_3 are constant. By substituting these equations into system (14)–(17) one obtains

$$\left(\frac{d\eta^4}{dT} - \frac{2}{3}\alpha_2 - \frac{4}{3}\alpha_1 \right) \frac{\partial p}{\partial T} + \alpha_1 \rho \frac{\partial^2 p}{\partial \rho \partial T} + \eta^4 \frac{\partial^2 p}{\partial T^2} = 0 \quad (18)$$

$$-\frac{2}{3}\alpha_2 - \frac{1}{3} + \alpha_1 \rho - \ln \frac{\partial p}{\partial \rho} + \eta^4 \ln \frac{\partial p}{\partial T} = 0 \quad (19)$$

$$\frac{4}{3}\alpha_1 - \frac{1}{3}\alpha_2 + \alpha_3 - \frac{d\eta^4}{dT} + \alpha_1 \frac{\rho}{\kappa} \frac{\partial \kappa}{\partial \rho} + \left(-\frac{3}{T} + \frac{1}{T} \frac{\partial \kappa}{\partial T} \right) = 0 \quad (20)$$

$$(\alpha_3 - \alpha_2)\epsilon = \alpha_1 \rho \frac{\partial \epsilon}{\partial T \rho} + \eta^4 \frac{\partial \epsilon}{\partial T}. \quad (21)$$

Equations (20) and (21) imply that the opacity coefficient κ and the energy generation rate ϵ are determined by the property of quasi-homologous temperature transformation generated by the component $\eta^4(T)\partial/\partial T$. This gives us the following solutions:

$$\phi \left\{ \rho \exp \left(-\alpha_1 \int_{T_0}^T \frac{dt}{\eta^4} \right), \kappa \exp \left[\int_{T_0}^T \frac{dt}{\eta^4} \left(\frac{d\eta^4}{dt} - \frac{3\eta^4}{t} + \frac{\alpha_2}{3} - \frac{4\alpha_1}{3} - \alpha_3 \right) \right] \right\} = 0 \quad (22)$$

$$\psi \left\{ \rho \exp \left(-\alpha_1 \int_{T_0}^T \frac{dt}{\eta^4(t)} \right), \rho \epsilon^{\alpha_1/(\alpha_3 - \alpha_2)} \right\} = 0 \quad (23)$$

where ϕ, ψ are arbitrary functions. Instead of form (22), equivalent and useful forms of representation of solutions (22) and (23) can be used in the further analysis [16], namely

$$\kappa(\rho, T) = \exp \left[\int_{T_0}^T \frac{dt}{\eta^4} \left(\frac{d\eta^4}{dt} - \frac{3\eta^4}{t} + \frac{\alpha_2}{3} - \frac{4\alpha_1}{3} - \alpha_3 \right) \right] f \left(\frac{1}{\rho} \exp \left(\alpha_1 \int_{T_0}^T \frac{dt}{\eta^4} \right) \right) \quad (24)$$

$$\epsilon(\rho, T) = \rho^{(\alpha_3 - \alpha_2)/\alpha_1} g \left(\rho^{-(\alpha_3 - \alpha_2)/\alpha_1} \exp \left((\alpha_3 - \alpha_2) \int_{T_0}^T \frac{dt}{\eta^4} \right) \right) \quad (25)$$

where f and g are arbitrary functions.

It still remains to solve equations (18) and (19). Since we have some freedom in choosing the equation $\eta^4(T)$, and the function $p(\rho, T)$ is never precisely known, we look for those equations of state $p = p(\rho, T)$ that are enforced by the transformations generated by operator (2). It is easy to check by using equations (18) and (19) that $p = p(\rho, T)$ satisfies the continuity condition $\partial^2 p / \partial T \partial \rho$ which in fact is a consistency condition for (18) and (19). Depending on the hyperbolic or parabolic character of equations (18) and (19) four cases can be distinguished. Let us discuss the solutions for these cases.

3.1. Case I

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + \eta^4(T) \frac{\partial}{\partial T}.$$

The standard method of reduction to the canonical form, when applied to equation (19), gives

$$\frac{\partial^2 p(x, T)}{\partial x \partial T} = \frac{4\alpha_1/3 + 2\alpha_2/3}{\eta^4(T)} \frac{\partial p}{\partial x} \quad (26)$$

where

$$x = \rho \exp\left(-\alpha_1 \int_{T_0}^T \frac{dt}{\eta^4(t)}\right).$$

In this case, the general solution of equation (26) assumes the form [16]

$$p(x, T) = h(T) + \int_{x_0}^x g(\zeta) d\zeta \exp\left(\int_{T_0}^T \frac{4\alpha_1/3 + 2\alpha_2/3}{\eta^4(T)}\right) \quad (27)$$

where $h(T)$ is of C^2 class and $g(\zeta)$ is of C^1 class of differentiability.

By substituting the general solution (27) into equation (18) we obtain the additional condition for $h(T)$

$$\eta^4(T) h''(T) + \left(\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 - \frac{d\eta^4}{dT}\right) h'(T) = 0. \quad (28)$$

The solution of this equation is

$$h(T) = C_1 \int^T \exp\left(\int^\tau \left(\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2\right) \frac{dt}{d\eta}\right) \frac{d\tau}{\eta^4(\tau)} + C_2 \quad (29)$$

where C_1 and C_2 are constants.

This solution implies that in equation (27) there is still the freedom in choosing a function $g = g(\zeta)$.

3.2. Case II

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + \eta^4(T) \frac{\partial}{\partial T}.$$

By proceeding in the same way as in the previous case we obtain

$$p(\rho, T) = h(T) + \int^\rho g(\rho') \exp\left(\int^T \frac{2\alpha_2}{3\eta^4(t)} dt\right) d\rho' \quad (30)$$

where

$$h(T) = C_1 \int^T \exp\left(\frac{2}{3}\alpha_2 \int^\tau \frac{dt}{\eta^4(t)}\right) \frac{d\tau}{\eta^4(\tau)} + C_2 \quad (31)$$

where h, g satisfy differentiability conditions as in the previous case and C_1, C_2 are constants.

3.3. Case III

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L}.$$

Similar calculations give

$$p(\rho, T) = h(\rho) + \int^T \rho^{4/3+2\alpha_2/3\alpha_1} g(t) dt \quad (32)$$

where

$$h(T) = C_1 \rho^{4/3+2\alpha_2/3\alpha_1} + C_2 \quad (33)$$

where h, g satisfy the differentiability conditions as previously and C_1, C_2 are constants.

3.4. Case IV

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L}.$$

In this case we have the following solutions:

$$p = p(\rho, T) \quad \alpha_2 = 0. \tag{34}$$

We can construct finite transformations and group invariants. In case I, for example, we obtain four independent invariants L_0, M_0, r_0 and ρ_0

$$\begin{aligned} L_0 &= L \exp\left(-\alpha_3 \int^T \frac{dt}{\eta^4(t)}\right) & M_0 &= M \exp\left(-\alpha_2 \int^T \frac{dt}{\eta^4(t)}\right) \\ r_0 &= r \exp\left(\frac{\alpha_1 - \alpha_2}{3} \int^T \frac{dt}{\eta^4(t)}\right) & \rho_0 &= \rho \exp\left(-\alpha_1 \int^T \frac{dt}{\eta^4(t)}\right). \end{aligned} \tag{35}$$

By using these invariants, we can construct the new families of solutions, for instance

(i) if $L(T)$ is the solution of (12), (8), (9) and (10), then also

$$L(E(T)) \exp\left(-\alpha_3 \int_{E(T_0)}^{E(T)} \frac{dt}{\eta^4(t)}\right)$$

is the solution, where $E(T)$ is a finite transformation of T , given from the solution $\bar{T} = E(T)$ of equation $d\bar{T}/d\tau = \eta^4(\bar{T})$ with the initial condition $\bar{T}(\tau = 0) = T$.

(ii) if $M(T)$ is the solution of (12), (8), (9) and (10), then also

$$M(E(T)) \exp\left(-\alpha_2 \int_{E(T_0)}^{E(T)} \frac{dt}{\eta^4(t)}\right)$$

is the solution.

4. Homologous symmetry transformations of structure equations with radiative transport

It is well known that equations (12), (8), (9) and (10) admit similarity symmetries for the equation of state of an ideal gas, $p \sim \rho T$. This fact induces a certain class of homologous solutions to the system. Some classical results were obtained by Stromgren [14].

In order to investigate how general our results are, let us assume the rescaling symmetry $\eta^4(T) = \alpha_4 T$. Then, in case I, for instance, the following equation of state is enforced:

$$p(x, T) = \frac{C_1 T^{(4\alpha_1 + 2\alpha_2)/3\alpha_4}}{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 + \alpha_4} + \int_{x_0}^x g(\zeta) d\zeta \frac{T^{1+(4\alpha_1 + 2\alpha_2)/3\alpha_4}}{\frac{4}{3}\alpha_1 + \frac{2}{3}\alpha_2 + \alpha_4}. \tag{36}$$

One should note that equation (36) contains the following form of the equation of state:

$$p = a\rho^A + bT^B + c\rho^C T^D.$$

The infinitesimal operator corresponding to the homologous transformations takes the form

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + \eta^4(T) \frac{\partial}{\partial T}. \tag{37}$$

The operator given by equation (37) has four independent invariants, for instance

$$J_1 = \rho r^{3\alpha_1/(\alpha_1 - \alpha_2)} \quad J_2 = \rho M^{-\alpha_1/\alpha_2} \quad J_3 = LM^{\alpha_3/\alpha_2} \quad J_4 = LT^{-\alpha_3/\alpha_4}.$$

By using these invariants we can arrive at various homology theorems, for example the theorem associated with J_1 : if $\rho(r)$ is the solution of (12) and (8)–(10), then $\rho\{r \exp[(\alpha_2 - \alpha_1)/3]\} \exp(-\alpha_1)$ is also the solution.

From J_3 we obtain $L \sim M^{\alpha_3/\alpha_2}$ that corresponds to the well-known Eddington mass–luminosity dependence. Infinitesimal operator (37) generates a Lie algebra spanned by the basic operators

$$X_1 = -\frac{1}{3}r \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} \quad X_2 = \frac{1}{3}r \frac{\partial}{\partial r} + M \frac{\partial}{\partial M} \quad X_3 = L \frac{\partial}{\partial L} \quad X_4 = T \frac{\partial}{\partial T}. \quad (38)$$

We now briefly present the basis operators for some particular cases that are important from the physical point of view.

(i) Photon gas, $p \sim T^4$

$$\begin{aligned} X_1 &= -\frac{r}{3} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \frac{T}{3} \frac{\partial}{\partial T} & X_2 &= \frac{r}{3} \frac{\partial}{\partial r} + M \frac{\partial}{\partial M} + \frac{T}{6} \frac{\partial}{\partial T} \\ X_3 &= L \frac{\partial}{\partial L} & X_4 &= 0. \end{aligned} \quad (39)$$

(ii) Ideal gas, $p \sim \rho T$

$$\begin{aligned} X_1 &= -\frac{r}{3} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \frac{T}{3} \frac{\partial}{\partial T} & X_2 &= \frac{r}{3} \frac{\partial}{\partial r} + M \frac{\partial}{\partial M} + \frac{T}{6} \frac{\partial}{\partial T} \\ X_3 &= L \frac{\partial}{\partial L} & X_4 &= 0. \end{aligned} \quad (40)$$

(iii) Degenerate gas, $p \sim \rho^{5/3}$

$$X_1 = -\frac{r}{6} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \frac{M}{2} \frac{\partial}{\partial M} \quad X_2 = 0 \quad X_3 = L \frac{\partial}{\partial L} \quad X_4 = M \frac{\partial}{\partial M}. \quad (41)$$

(iv) Relativistic degenerate electron gas, $p \sim \rho^{4/3}$

$$X_1 = -\frac{r}{3} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} \quad X_2 = L \frac{\partial}{\partial L} \quad X_3 = 0 \quad X_4 = M \frac{\partial}{\partial M}. \quad (42)$$

(v) Ideal and photon gas

$$X_1 = -\frac{r}{3} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \frac{T}{3} \frac{\partial}{\partial T} \quad X_2 = 0 \quad X_3 = L \frac{\partial}{\partial L} \quad X_4 = 0. \quad (43)$$

It is interesting to note that the Eddington mass–luminosity dependence is not satisfied any longer, as there are no non-trivial invariants associated with M .

(vi) Ideal and degenerate gas

$$X_1 = -\frac{r}{6} \frac{\partial}{\partial r} + \rho \frac{\partial}{\partial \rho} + \frac{M}{2} \frac{\partial}{\partial M} + \frac{2T}{3} \frac{\partial}{\partial T} \quad X_2 = 0 \quad X_3 = L \frac{\partial}{\partial L} \quad X_4 = 0. \quad (44)$$

5. Quasi-homologous stars with convective transport

We start the analysis of stars with convection from equations (7)–(9) and (11) which describe the Newtonian static star with convective transport energy. The convective transport of energy means an exchange of energy between hotter and cooler layers through the exchange of macroscopic mass elements, the hotter of which move upwards while the cooler ones descend.

By analogy with the previous analysis instead of equation (11), it would be useful to operate the new equation that can be constructed if we assume the form of the equation of state $p = p(\rho, T)$. Then we obtain

$$\frac{d\rho}{dr} = \left(\frac{\partial p}{\partial \rho}\right)^{-1} \left(-1 + \Gamma(\rho, T) \frac{T}{p} \frac{\partial p}{\partial T}\right) \frac{GM\rho}{r^2} = f \left(\frac{\partial p}{\partial \rho}\right)^{-1} \frac{GM}{r^2}. \tag{45}$$

We search for the symmetry operator in form (13) for system (7)–(9) and (11). After extension of operator (13) on the first derivatives we obtain the equations admissible for this operator X' in the form

$$\begin{aligned} \frac{d\eta^1(\rho)}{d\rho} - \frac{d\xi(r)}{dr} = X(\ln f^1) = & -\frac{2\xi(r)}{r} + \frac{\eta^1(\rho)}{\rho} - \eta^4 \frac{\partial}{\partial T} \ln \left(\frac{\partial p}{\partial \rho}\right) + \frac{\eta^2(M)}{M} \\ & - \eta^1 \frac{\partial}{\partial \rho} \ln \left(\frac{\partial p}{\partial \rho}\right) + \frac{\eta^1(\rho)}{f} \left(\frac{T}{p} \frac{\partial p}{\partial T} \frac{\partial \Gamma}{\partial \rho} + \Gamma \frac{T}{p} \frac{\partial^2 p}{\partial T \partial \rho} - \frac{\Gamma T}{p^2} \frac{\partial p}{\partial T} \frac{\partial p}{\partial \rho}\right) \\ & + \frac{\eta^4(T)}{f} \left[\frac{T}{p} \frac{\partial \Gamma}{\partial T} \frac{\partial p}{\partial T} + \frac{\Gamma}{p} \frac{\partial p}{\partial T} - \frac{\Gamma T}{p^2} \left(\frac{\partial p}{\partial T}\right)^2 + \frac{\Gamma T}{p} \frac{\partial^2 p}{\partial T^2}\right] \end{aligned} \tag{46}$$

$$\frac{d\eta^2(M)}{dM} - \frac{d\xi(r)}{dr} = X(\ln f^2) = \frac{2\xi(r)}{r} + \frac{\eta^1(\rho)}{\rho} \tag{47}$$

$$\frac{d\eta^3(L)}{dL} - \frac{d\xi(r)}{dr} = X(\ln f^3) = \frac{2\xi(r)}{r} + \frac{\eta^1(\rho)}{\rho} + \frac{\eta^1}{\epsilon} \frac{\partial \epsilon}{\partial \rho} + \frac{\eta^4}{\epsilon} \frac{\partial \epsilon}{\partial T} \tag{48}$$

$$\begin{aligned} \frac{d\eta^4(T)}{dT} - \frac{d\xi(r)}{dr} = X(\ln f^4) = & -\frac{2\xi(r)}{r} + \frac{\eta^2(M)}{M} + \frac{\eta^1}{\rho} + \frac{\eta^1}{\Gamma} \frac{\partial \Gamma}{\partial \rho} \\ & - \frac{\eta^1}{p} \frac{\partial p}{\partial \rho} + 2 \frac{\eta^4}{\Gamma} \frac{\partial \Gamma}{\partial T} + \frac{\eta^4}{T} - \frac{\eta^4}{p} \frac{\partial p}{\partial T}. \end{aligned} \tag{49}$$

From equation (48) we obtain $\eta^3(L) = \alpha_3 L$, where α_3 is constant and equation (47) gives us the linear relation $\eta^2(M) = \alpha_2 M + \beta_2$ and $\eta^1(\rho) = \alpha_1 \rho$, where $\alpha_1, \alpha_2, \beta_2$ are constant. Then equation (48) determines the constraint equation in the form

$$\frac{d\xi(r)}{dr} + \frac{2\xi(r)}{r} = \alpha_2 - \alpha_1 \tag{50}$$

which can be easily integrated and we obtain

$$\xi(r) = \frac{\alpha_2 - \alpha_1}{3} r + \frac{C}{r}. \tag{51}$$

After substituting the above solution into equations (46) and (49) we finally obtain that constant C in (51) must vanish.

Then from equation (48) we obtain

$$\alpha_1 \rho \frac{\partial \epsilon}{\partial \rho} + \eta^4(T) \frac{\partial \epsilon}{\partial T} - (\alpha_3 - \alpha_2) \epsilon = 0 \tag{52}$$

which can be integrated by using the standard characteristic method. The invariants are determined from the characteristic equation

$$\frac{d\rho}{\alpha_1 \rho} = \frac{dT}{\eta^4(T)} = \frac{d\epsilon}{(\alpha_3 - \alpha_2) \epsilon}$$

where we assume that $\alpha_3 - \alpha_2 \neq 0$ then

$$C_1 = \frac{\rho^{(\alpha_3 - \alpha_2)/\alpha_1}}{\epsilon} \quad C_2 = \frac{\exp\left[(\alpha_3 - \alpha_2) \int^T \frac{dT}{\eta^4(T)}\right]}{\epsilon}. \tag{53}$$

It is well known that the general solution of equation (52) can be given in the form of any function ψ of its invariants, i.e.

$$\psi \left(\frac{\rho^{(\alpha_3 - \alpha_2)/\alpha_1}}{\epsilon}, \frac{\exp \left[(\alpha_3 - \alpha_2) \int^T \frac{dT}{\eta^4(T)} \right]}{\epsilon} \right) = 0. \quad (54)$$

Following Sundman there is the equivalent form of the general solution in which the production energy function can be given in the exact form, namely

$$\epsilon = \rho^{(\alpha_3 - \alpha_2)/\alpha_1} g \left(\frac{\exp \left[(\alpha_3 - \alpha_2) \int^T \frac{dT}{\eta^4(T)} \right]}{\rho^{(\alpha_3 - \alpha_2)/\alpha_1}} \right). \quad (55)$$

From (46) or (49) we obtain that

$$\eta^3(M) = \alpha_2 M + \beta_2$$

and $\beta_2 = 0$.

Therefore all components of operator X except $\eta^4(T)$ are determined

$$\xi(r) = \frac{\alpha_2 - \alpha_1}{3} r \quad \eta^1(\rho) = \alpha_1 \rho \quad \eta^2(M) = \alpha_2 M \quad \eta^3(L) = \alpha_3 L. \quad (56)$$

In our further analysis we assume that $\Gamma = \Gamma_0 = \text{const}$. This assumption simplifies our calculations but we must remember that in general $\Gamma = d \ln T / d \ln p$. (If the energy transport is due to radiation and conduction then Γ must be replaced by $\Gamma_{\text{rad}} = 3\kappa L p / 16\pi a c G m T^4$.) This property is destroyed if the material function, instead of being products of power p and T contains additive terms as in the general case with the equation of state. The simplest example is the addition of radiation to ideal gas such that $p = \mathcal{R} \rho T / \mu + a T^4 / 3 = p_g + p_r$. No strict homology (quasi-homology) relation is then possible. But one can try to make the enlarged system of equations in which the equation for $d\beta/dr$ is added ($\beta = p_g/p$). The corresponding equation is

$$\frac{d\beta}{dr} = \beta \frac{d}{dr} \ln \left(\frac{\mathcal{R} \rho T}{\mu p} \right) = \beta \frac{d}{dr} \ln \left(\frac{\rho \Gamma}{p(\rho, T)} \right). \quad (57)$$

Now β is not constant and in equation (57) the relations determining $d\rho/dr$, dT/dr and dp/dr should be substituted. As a result we obtain that the enlarged system admits the homology relation.

Because we focus on the simplest possible case the problem with constant Γ is considered. For $\Gamma = \Gamma_0$, equation (46) assumes the form

$$\begin{aligned} \frac{\alpha_2 - \alpha_1}{3} &= \alpha_2 - \alpha_1 \rho \frac{\partial}{\partial \rho} \ln \left(\frac{\partial p}{\partial \rho} \right) - \eta^4 \frac{\partial}{\partial T} \left(\ln \frac{\partial p}{\partial \rho} \right) + \frac{\alpha_1 \rho}{f} \left(\frac{\Gamma_0 T}{p} \frac{\partial^2 p}{\partial \rho \partial T} - \frac{\Gamma_0 T}{p^2} \frac{\partial p}{\partial T} \frac{\partial p}{\partial \rho} \right) \\ &+ \frac{\eta^4(T)}{f} \left[\frac{\Gamma_0}{p} \frac{\partial p}{\partial T} - \frac{\Gamma_0 T}{p^2} \left(\frac{\partial p}{\partial T} \right)^2 + \frac{\Gamma_0 T}{p} \frac{\partial^2 p}{\partial T^2} \right]. \end{aligned} \quad (58)$$

Quite similarly we can write equation (49) in the constant Γ and we have

$$\frac{d\eta^4(T)}{dT} - \frac{\alpha_2 - \alpha_1}{3} = -\frac{2}{3}(\alpha_2 - \alpha_1) + \alpha_1 + \alpha_2 + \frac{\eta^4(T)}{T} - \frac{\alpha_1 \rho}{p} \frac{\partial p}{\partial \rho} \quad (59)$$

or

$$\alpha_1 \rho \frac{\partial p}{\partial \rho} + \eta^4(T) \frac{\partial p}{\partial T} = \left(-\frac{d\eta^4}{dT} + \frac{\eta^4}{T} + \frac{4\alpha_1 + 2\alpha_2}{3} \right) p.$$

The solution of equation (59) can be given in the form of an arbitrary function ψ of its invariants, namely

$$\psi \left(\rho^{-1} \exp \left(\int^T \frac{\alpha_1}{\eta^4} dt \right), p^{-1} \exp \left(\int^T \frac{dt}{\eta^4} \left(-\frac{d\eta^4}{dt} + \frac{\eta^4}{t} + \frac{4\alpha_1 + 2\alpha_2}{3} \right) \right) \right) = 0 \tag{60}$$

or

$$p(\rho, T) = \exp \left(\int^T \frac{dt}{\eta^4} \left(-\frac{d\eta^4}{dt} + \frac{\eta^4}{t} + \frac{4\alpha_1 + 2\alpha_2}{3} \right) \right) g \left(\rho^{-1} \exp \left(\int^T \frac{\alpha_1}{\eta^4} dt \right) \right). \tag{61}$$

After substituting the corresponding terms into (59) from the exact formula for $p(\rho, T)$, the following condition must be fulfilled

$$\begin{aligned} \frac{d\eta_4}{dT} - \frac{\eta_4}{T} - \Gamma_0 T \left(-\frac{d\eta_4}{dT} + \frac{\eta_4}{T} + \frac{4\alpha_1 + 2\alpha_2}{3} \right) \frac{1}{\eta_4} \frac{d\eta_4}{dT} \\ - \Gamma_0 T \eta_4 \frac{\partial}{\partial T} \left[\left(-\frac{d\eta_4}{dT} + \frac{\eta_4}{T} + \frac{4\alpha_1 + 2\alpha_2}{3} \right) \frac{1}{\eta_4} \right] = 0. \end{aligned} \tag{62}$$

The above relation reduces to the simpler form

$$\frac{d^2\eta_4}{dT^2} + \frac{1 - \Gamma_0}{\Gamma_0} \frac{1}{T} \left(\frac{d\eta_4}{dT} - \eta_4 \right) = 0. \tag{63}$$

The solution of the above equation for any Γ_0 can be obtained if we substitute $\eta_4 \approx T^x$. Then

$$x_1 = 1 \quad x_2 = \frac{\Gamma_0}{\Gamma_0 - 1} \tag{64}$$

and finally we have

$$\eta_4 = C_1 T + C_2 T^{(\Gamma_0-1)/\Gamma_0} \tag{65}$$

where C_1 and C_2 are constants.

Therefore, we find that for the general form of the equation of state, the quasi-homology symmetry is admitted by the stellar structure, e.g. with convective transport, i.e., the operator of this symmetry is

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + C_1 T + C_2 T^{(\Gamma_0-1)/\Gamma_0} \frac{\partial}{\partial T}. \tag{66}$$

As a special case for $C_2 = 0$ we find that homologous symmetry is obvious and $p = T^{(4\alpha_1 - 2\alpha_2)/3\alpha_4} g\left(\frac{T^{\alpha_1/\alpha_4}}{g}\right)$.

The non-trivial class of quasi-homologous (and homologous) relations can be obtained from the invariants. For example, from the invariant J_1 such that

$$\frac{dL}{\alpha_3 L} = \frac{dT}{C_1 T + C_2 T^{(\Gamma_0-1)/\Gamma_0}} \quad J_1 = L(T) \exp \left(-\alpha_3 \int^T \frac{d\tau}{C_1 \tau + C_2 \tau^{(\Gamma_0-1)/\Gamma_0}} \right). \tag{67}$$

It is possible to obtain a new class of solutions from the known ones which can be treated as a better reconstruction of the main sequence in the HR diagram by using only theoretical assumptions.

6. Quasi-homologous stars with convective transport

As the simplest example let us consider a star filled by radiation, i.e., $p(\rho, T) = aT^4$, $\Gamma = \Gamma_0 = \frac{1}{4}$, $g = 1$, then

$$X = \frac{\alpha_2 - \alpha_1}{3} r \frac{\partial}{\partial r} + \alpha_1 \rho \frac{\partial}{\partial \rho} + \alpha_2 M \frac{\partial}{\partial M} + \alpha_3 L \frac{\partial}{\partial L} + (C_1 T + C_2 T^{-3}) \frac{\partial}{\partial T}. \tag{68}$$

The important quasi-homologous theorem useful in the generation of a solution from the known ones can be formulated in the following way. If $L(T)$ is a solution of the stellar structure equation in a convective equilibrium, then

$$L(T') \exp\left(-\alpha_3 \int^{\tau} \frac{d\tau}{C_1\tau + C_2\tau^{-3}}\right) = \text{const} \quad (69)$$

is also a solution, where $T' = E(T)$ is a finite transformation which is determined as a solution of equation

$$\int^{\tau} \frac{dT}{C_1T + C_2T^{-3}} = \tau$$

and $T'(\tau = 0) = T$.

After integration we obtain the exact form of the finite transformation

$$T' = E(T): C_1(T')^4 + C_2 = (C_1T^4 + C_2) e^{4C_1\tau}.$$

This procedure can be repeated in the more general case of constant Γ_0 . Then we obtain

$$T' = E(T): C_1(T')^{1/\Gamma_0} + C_2 = (C_1T^{1/\Gamma_0} + C_2) e^{C_1\tau/\Gamma_0}.$$

7. Conclusion

We investigated symmetries for the equations of state of ideal gas, polytropic gas, photon gas, degenerated gas, as well as some mixed cases, and found the most general form of quasi-homologous transformation admitted by the system of equations describing Newtonian static stars. In considering the invariants of the symmetry groups we noted that they correspond to the relations describing physical characteristics of the stars, such as the Hertzsprung–Russell diagram or the mass–luminosity relation. The group invariants make possible the construction of quasi-homologous theorems analogous to the Chandrasekhar homologous theorems, which provide a prescription for deriving new classes of solutions from the present known solutions.

We characterized, by computing infinitesimal operators, the structure of the group admissible by a system of equations describing Newtonian static stars in radiative and convective equilibrium. We have shown that, in the most general case, the equations admit an infinite parameter group of quasi-homologous transformations. These symmetries enforce appropriate equations of state. In the particular case of a five parameter homologous group the Stromgren results are recovered.

The equations of state (36) and (61) and parametrization of energy production and capacity are very general. They contain both physical and non-physical situations. Our results suggest that group invariants do not always have purely mathematical sense, but that their existence is closely associated with basic empirical facts such as the Hertzsprung–Russell diagram or the Eddington mass–luminosity dependence. The fact that the main sequence in the Hertzsprung–Russell diagram can be reconstructed from the invariants proved the effectiveness of the group analysis of fundamental equations of astrophysics. We indicated this in two examples on the main sequence models and radiative and convective transport. The main result is that except for classical homology there is another type of quasi-homologous symmetry which can be applied to certain red giants (see [17, section 32.2]).

Due to the existence of a new ‘similarity’ between different solutions, the quasi-homology relation offers qualitative but helpful indication for interpreting or predicting the numerical solutions.

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References

- [1] Chandrasekhar S 1957 *An Introduction to the Study of Stellar Structure* (New York: Dover)
- [2] Rudzki A 1902 *Bull. Astrophys.* **19** 134
- [3] Eddington A S 1959 *The Internal Constitution of the Stars* (New York: Dover)
- [4] Schwarzschild M 1958 *Structure and Evolution of the Stars* (Princeton, NJ: Princeton University Press)
- [5] Biesiada M, Golda Z and Szydłowski M 1987 *J. Phys. A: Math. Gen.* **20** 1313
- [6] Biesiada M and Szydłowski M 1988 *J. Phys. A: Math. Gen.* **21** 3409
- [7] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (New York: Academic)
- [8] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (New York: Springer)
- [9] Stefani H 1989 *Differential Equations: Their Solutions Using Symmetries* (New York: Cambridge University Press)
- [10] Olver P J 1993 *Applications of Lie Groups to Differential Equations* 2nd edn (New York: Springer)
- [11] Olver P J 1995 *Equivalence, Invariants and Symmetry* (New York: Cambridge University Press)
- [12] Barenblatt G I 1996 *Scaling, Self-Similarity and Intermediate Asymptotics* (New York: Cambridge University Press)
- [13] Collins C B 1977 *J. Math. Phys.* **18** 1374
- [14] Stromgren B 1937 *Ery. Exapt. Naturw.* **16** 465
- [15] Bluman G W and Cole J D 1974 *Similarity Methods for Differential Equations* (New York: Springer)
- [16] Sneddon I N 1957 *Elements of Partial Differential Equations* (New York: McGraw-Hill)
- [17] Kippenhahn R and Weigert A 1990 *Stellar Structure and Evolution* (Berlin: Springer)